Workshop: Differentiation – the rules

Topics Covered:
- Chain rule
- Product rule
- Quotient rule
- Combinations

Chain Rule

To differentiate composite functions we have to use the Chain rule.

Composite functions are functions of a function.

If we have y = f(t) and t = g(x), then the derivative of y with respect to x is,

y is a function in terms of t

t is a function in terms of x

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Example 1: Differentiate $y = (3x - 1)^2$ with respect to x

Let $t = 3x - 1$ $\qquad$ $y = t^2$

$\frac{dt}{dx} = 3$ $\qquad$ $\frac{dy}{dt} = 2t$

Using the chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$\frac{dy}{dx} = 3 \times 2t = 6t = 6(3x - 1)$$

Example 2: Differentiate $y = \sqrt{5x + 9}$ with respect to x

$$y = \sqrt{5x + 9} = (5x + 9)^{\frac{1}{2}}$$

Let $t = 5x + 9$ $\qquad$ $y = t^{½}$

$\frac{dt}{dx} = 5$ $\qquad$ $\frac{dy}{dt} = ½t^{-½}$

Using the chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$\frac{dy}{dx} = 5 \times ½t^{-½} = \frac{5}{2}t^{-\frac{1}{2}} = \frac{5}{2}(5x+9)^{-\frac{1}{2}} = \frac{5}{2(5x+9)^{\frac{1}{2}}}$$

Example 3: Differentiate y = sin $(2x^3 + 9x)$ with respect to x.

Let $t = 2x^3 + 9x$ $\qquad$ $y = \sin t$

$\frac{dt}{dx} = 6x^2 + 9$ $\qquad$ $\frac{dy}{dt} = \cos t$

Using the Chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$\frac{dy}{dx} = \cos(t) \times (6x^2 + 9) = (6x^2 + 9)\cos(2x^3 + 9x)$$

Questions (Chain rule)**:**

Differentiate the following functions with respect to x.

1. $y = (3x^2 + 2x)^5$
2. $y = \cos(5x^3 + 6)$
3. $y = e^{x^3+2x^2}$
4. $y = \ln(x^4 + 2x)$

(Solutions on page 8)

The Product Rule

Consider y = uv, where u and v are functions of x, then

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Example 1: Differentiate $y = 7xe^{2x}$

Let $u = 7x$ $\quad v = e^{2x}$

$\frac{du}{dx} = 7$ $\quad \frac{dv}{dx} = 2e^{2x}$

Using the product rule: $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$$\frac{dy}{dx} = 7e^{2x} + 14xe^{2x} = 7e^{2x}(1 + 2x)$$

Example 2: Differentiate $y = x^3\sin(3x)$

Let $u = x^3$ $\quad v = \sin(3x)$

$\frac{du}{dx} = 3x^2$ $\quad \frac{dv}{dx} = 3\cos(3x)$

Using the product rule: $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$$\frac{dy}{dx} = 3x^2\sin(3x) + 3x^3\cos(3x)$$

$$= 3x^2(\sin(3x) + x\cos(3x))$$

Example 3: Differentiate $y = e^{2x}\ln(5x)$

Let $u = e^{2x}$ $\quad v = \ln(5x)$

$\frac{du}{dx} = 2e^{2x}$ $\quad \frac{dv}{dx} = \frac{1}{x}$

Using the product rule: $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$$\frac{dy}{dx} = 2e^{2x}\ln(5x) + \frac{e^{2x}}{x}$$

$$= e^{2x}\left(2\ln(5x) + \frac{1}{x}\right)$$

Questions (Produce rule)**:**

Differentiate the following functions with respect to x:

1. $y = 5x\ln(6x)$
2. $y = e^{5x}(4x + 2)$
3. $y = 3x^5\cos(4x)$
4. $y = e^{4x}\ln(7x)$

(Solutions on page 9)

The Quotient Rule

Consider $y = \frac{u}{v}$, where u and v are functions of x, then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 1: Differentiate $y = \frac{x^2 - 1}{x^2 + 1}$

$u = x^2 - 1$ $\qquad$ $v = x^2 + 1$

$\frac{du}{dx} = 2x$ $\qquad$ $\frac{dv}{dx} = 2x$

Using the quotient rule: $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2}$$

$$= \frac{2x\left[\left(x^2+1\right)-\left(x^2-1\right)\right]}{\left(x^2+1\right)^2}$$

$$= \frac{2x\left(x^2+1-x^2+1\right)}{\left(x^2+1\right)^2}$$

$$= \frac{2x(2)}{\left(x^2+1\right)^2}$$

$$= \frac{4x}{\left(x^2+1\right)^2}$$

Example 2: Differentiate $y = \frac{\ln x}{x^2+5x}$

$u = \ln(x)$ $\qquad v = x^2 + 5x$

$\frac{du}{dx} = \frac{1}{x}$ $\qquad \frac{dv}{dx} = 2x + 5$

Using the quotient rule: $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\frac{1}{x}\left(x^2+5x\right)-(2x+5)\ln x}{\left(x^2+5x\right)^2}$$

$$= \frac{(x+5)-(2x+5)\ln x}{\left(x^2+5x\right)^2}$$

Example 3: Differentiate $y = \frac{5e^x-1}{3e^x+2x}$

$u = 5e^x - 1$ $\qquad v = 3e^x + 2x$

$\frac{du}{dx} = 5e^x$ $\qquad \frac{dv}{dx} = 3e^x + 2$

Using the quotient rule: $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{5e^x(3e^x+2x)-(5e^x-1)(3e^x+2)}{(3e^x+2x)^2}$$

$$= \frac{15e^{2x}+10xe^x-(15e^{2x}+10e^x-3e^x-2)}{(3e^x+2x)^2}$$

$$= \frac{15e^{2x}+10xe^x-15e^{2x}-7e^x+2}{(3e^x+2x)^2}$$

$$= \frac{10xe^x-7e^x+2}{(3e^x+2x)^2}$$

Questions (Quotient rule)**:**

Differentiate the following functions with respect to x:

1. $y = \dfrac{x^2}{1-x}$
2. $y = \dfrac{4x^3}{\cos x}$
3. $y = \dfrac{\ln(x)}{x^2-1}$
4. $y = \dfrac{e^{2x}}{\sin x+\cos x}$

(Solutions on page 10)

Combinations

Sometimes you may need to use a combination of the rules to differentiate a function.

Example 1: Differentiate $y = x^5(x+3)^4$

Solution:

Let $u = x^5$ $\qquad v = (x+3)^4$

$\dfrac{du}{dx} = 5x^4$ $\qquad \dfrac{dv}{dx} = 4(x+3)^3$

Using the chain rule:

$t = x + 3 \qquad v = t^4$

$\dfrac{dt}{dx} = 1 \qquad \dfrac{dv}{dt} = 4t^3$

Using the chain rule:

$$\frac{dv}{dx} = \frac{dt}{dx}\times\frac{dv}{dt}$$

$= 1 \times 4t^3$

$= 4(x+3)^3$

Using the product rule: $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$$\frac{dy}{dx} = 5x^4(x + 3)^4 + 4x^5(x + 3)^3$$
$$= x^4(x + 3)^3[5(x + 3) + 4x]$$
$$= x^4(x + 3)^3[5x + 15 + 4x]$$
$$= x^4(x + 3)^3(9x + 15)$$

Example 2: Differentiate y = $\frac{\ln(x^2+1)}{x^2-1}$

Solution:

$u = \ln(x^2 + 1) \qquad v = x^2 - 1$

$\frac{du}{dx} = \frac{2x}{x^2+1} \qquad \frac{dv}{dx} = 2x$

Using the chain rule:
$t = x^2 + 1 \qquad u = \ln t$
$\frac{dt}{dx} = 2x \qquad \frac{du}{dt} = \frac{1}{t}$

Using the chain rule:
$$\frac{du}{dx} = \frac{dt}{dx} \times \frac{du}{dt}$$
$$= 2x \times \frac{1}{t}$$
$$= \frac{2x}{t} = \frac{2x}{x^2+1}$$

Using the quotient rule: $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(x^2 - 1)\left(\frac{2x}{x^2+1}\right) - \ln(x^2+1)\times(2x)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{2x(x^2 - 1) - 2x(x^2+1)\ln(x^2+1)}{(x^2+1)(x^2-1)^2}$$

Questions (Combinations)**:**

Differentiate the following functions with respect to x:

1. $y = (2x^2 + 3)(3x + 4)^3$
2. $y = \frac{x^4}{(x+1)^2}$

(Solutions on page 12)

Solutions (Chain rule)**:**

1. $y = (3x^2 + 2x)^5$

   Let $t = 3x^2 + 2x$ $\qquad y = t^5$

   $\frac{dt}{dx} = 6x + 2$ $\qquad \frac{dy}{dt} = 5t^4$

   Using the chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

   $\frac{dy}{dx} = 5t^4 \times (6x + 2)$

   $= 5t^4(6x + 2) = 5(3x^2 + 2x)^4(6x + 2)$

2. $y = \cos(5x^3 + 6)$

   Let $t = 5x^3 + 6$ $\qquad y = \cos t$

   $\frac{dt}{dx} = 15x^2$ $\qquad \frac{dy}{dt} = -\sin t$

   Using the chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

   $\frac{dy}{dx} = -\sin t \times (15x^2) = -15x^2\sin(t) = -15x^2\sin(5x^3 + 6)$

3. $y = e^{x^3+2x^2}$

   Let $t = x^3 + 2x^2$ $\qquad y = e^t$

   $\frac{dt}{dx} = 3x^2 + 4x$ $\qquad \frac{dy}{dt} = e^t$

   Using the chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

   $\frac{dy}{dx} = (3x^2 + 4x)\,e^{x^3+2x^2}$

4. $y = \ln(x^4 + 2x)$

   Let $t = x^4 + 2x$ $\qquad y = \ln t$

   $\frac{dt}{dx} = 4x^3 + 2$ $\qquad \frac{dy}{dt} = \frac{1}{t}$

   Using the chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$\frac{dy}{dx} = (4x^3 + 2) \times \frac{1}{t}$$

$$= \frac{(4x^3+2)}{(x^4+2x)}$$

Solutions (Product rule)**:**

1. $y = 5x\ln(6x)$

   Let $u = 5x$ $\quad v = \ln(6x)$

   $\frac{du}{dx} = 5 \quad \frac{dv}{dx} = \frac{1}{x}$

   Using the product rule: $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

   $$\frac{dy}{dx} = 5\ln(x) + 5x\left(\frac{1}{x}\right)$$

   $$= 5\ln(x) + 5$$

   $$= 5(\ln(x) + 1)$$

2. $y = e^{5x}(4x + 2)$

   Let $u = e^{5x}$ $\quad v = 4x + 3$

   $\frac{du}{dx} = 5e^{5x} \quad \frac{dv}{dx} = 4$

   Using the product rule: $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

   $$\frac{dy}{dx} = 5e^{5x}(4x + 3) + 4e^{5x}$$

   $$= e^{5x}(5(4x + 3) + 4)$$

   $$= e^{5x}(20x + 15 + 4)$$

   $$= e^{5x}(20x + 19)$$

3. $y = 3x^5\cos(4x)$

   Let $u = 3x^5$ $\quad v = \cos(4x)$

   $\frac{du}{dx} = 15x^4 \quad \frac{dv}{dx} = -4\sin(4x)$

Using the product rule: $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$$\frac{dy}{dx} = 15x^4\cos(4x) - 12x^5\sin(4x)$$

$$= 3x^4[5\cos(4x) - 4x\sin(4x)]$$

4. $y = e^{4x}\ln(7x)$

Let $u = e^{4x}$ $\qquad v = \ln(7x)$

$\frac{du}{dx} = 4e^{4x}$ $\qquad \frac{dv}{dx} = \frac{1}{x}$

Using the product rule: $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$$\frac{dy}{dx} = 4e^{4x}\ln(7x) + \frac{e^{4x}}{x}$$

$$= e^{4x}\left(4\ln(7x) + \frac{1}{x}\right)$$

Solutions (Quotient rule)**:**

1. $y = \frac{x^2}{1-x}$

$u = x^2$ $\qquad v = 1 - x$

$\frac{du}{dx} = 2x$ $\qquad \frac{dv}{dx} = -1$

Using the quotient rule: $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{2x(1-x) - x^2(-1)}{(1-x)^2}$$

$$= \frac{2x - 2x^2 + x^2}{(1-x)^2}$$

$$= \frac{2x - x^2}{(1-x)^2}$$

$$= \frac{x(2-x)}{(1-x)^2}$$

2. $y = \frac{4x^3}{\cos x}$

$u = 4x^3$ $v = \cos(x)$

$\frac{du}{dx} = 12x^2$ $\frac{dv}{dx} = -\sin x$

Using the quotient rule: $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{12x^2\cos x + 4x^3\sin x}{(\cos x)^2}$$

$$= \frac{4x^3(3\cos x + x\sin x)}{\cos^2 x}$$

3. $y = \frac{\ln(x)}{x^2 - 1}$

$u = \ln(x)$ $v = x^2 - 1$

$\frac{du}{dx} = \frac{1}{x}$ $\frac{dv}{dx} = 2x$

Using the quotient rule: $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(x^2 - 1)\left(\frac{1}{x}\right) - 2x\ln(x)}{(x^2 - 1)^2}$$

$$\frac{dy}{dx} = \frac{(x - \frac{1}{x}) - 2x\ln(x)}{(x^2 - 1)^2}$$

$$\frac{dy}{dx} = \frac{x^2 - 1 - 2x^2\ln(x)}{x(x^2 - 1)^2}$$

4. $y = \dfrac{e^{2x}}{\sin x + \cos x}$

$u = e^{2x}$ $\qquad v = \sin(x) + \cos(x)$

$\dfrac{du}{dx} = 2e^{2x}$ $\qquad \dfrac{dv}{dx} = \cos(x) - \sin(x)$

Using the quotient rule: $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(\sin x + \cos x)(2e^{2x}) - (e^{2x})(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= \frac{e^{2x}[2(\sin x + \cos x) - (\cos x - \sin x)]}{(\sin x + \cos x)^2}$$

$$= \frac{e^{2x}(2\sin x + 2\cos x - \cos x + \sin x)}{(\sin x + \cos x)^2}$$

$$= \frac{e^{2x}(3\sin x + \cos x)}{(\sin x + \cos x)^2}$$

Solutions (Combinations)**:**

1. $y = (2x^2 + 3)(3x + 4)^3$

Let $u = (2x^2 + 3)$ $\qquad v = (3x + 4)^3$

$\dfrac{du}{dx} = 4x$ $\qquad \dfrac{dv}{dx} = 3(3x + 4)^2.3$

$= 9(3x + 4)^2$

Using the product rule: $\dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$

$$\frac{dy}{dx} = 4x(3x + 4)^3 + 9(2x^2 + 3)(3x + 4)^2$$

$$= (3x + 4)^2[4x(3x + 4) + 9(2x^2 + 3)]$$

$$= (3x + 4)^2[12x^2 + 16x + 18x^2 + 27]$$

$$= (3x + 4)^2[30x^2 + 16x + 27]$$

2. $y = \frac{x^4}{(x+1)^2}$

Using the quotient rule: $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$u = x^4$ $v = (x + 1)^2$

$\frac{du}{dx} = 4x^3$ $\frac{dv}{dx} = 2(x + 1)$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{4x^3(x+1)^2 - 2x^4(x+1)}{\left[(x+1)^2\right]^2}$$

$$= \frac{4x^3(x+1)^2 - 2x^4(x+1)}{(x+1)^4}$$

$$= \frac{4x^3(x+1) - 2x^4}{(x+1)^3}$$

$$= \frac{4x^4 + 4x^3 - 2x^4}{(x+1)^3}$$

$$= \frac{2x^4 + 4x^3}{(x+1)^3}$$

$$= \frac{2x^3(x+2)}{(x+1)^3}$$