Workshop: Introduction to Differentiation

Topics Covered:

- Power rule, constant-multiple rule, constant term
- Ln x, exponentials, sine and cosine
- Higher order derivatives
- Maxima, Minima and Points of Inflection

Differentiation of polynomials and algebraic functions

Power rule

When $\mathbf{y = x^n}$, where n is any real number

$$\frac{dy}{dx} = nx^{n-1}$$

Examples: $y = x^4$, $\frac{dy}{dx} = 4x^3$

$y = x^{-2}$, $\frac{dy}{dx} = -2x^{-3}$

Constant-multiple rule

When $\mathbf{y = cx^n}$, where c is a constant

$$\frac{dy}{dx} = cnx^{n-1}$$

Examples: $y = 2x^3$, $\frac{dy}{dx} = 6x^2$

$y = 5x^{-3}$, $\frac{dy}{dx} = -15x^{-4}$

Constant term

When $\mathbf{y = c}$, where c is a constant

$$\frac{dy}{dx} = 0$$

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Sum and difference rule

If $\mathbf{y = u \pm v}$, where u and v are functions of x,

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\mathbf{du}}{\mathbf{dx}} \pm \frac{\mathbf{dv}}{\mathbf{dx}}$$

Examples: $y = 3x^5 + x^6$, $\quad \frac{dy}{dx} = 15x^4 + 6x^5$

$y = 4x^5 - x^{-3}$, $\quad \frac{dy}{dx} = 20x^4 + 3x^{-4}$

Questions		Solutions	
1.	$y = x^8$	1.	$\frac{dy}{dx} = 8x^7$
2.	$y = x^{-5}$	2.	$\frac{dy}{dx} = -5x^{-6}$
3.	$y = 9x^2$	3.	$\frac{dy}{dx} = 18x$
4.	$y = 3x^3$	4.	$\frac{dy}{dx} = 9x^2$
5.	$y = 7x^{-2}$	5.	$\frac{dy}{dx} = -14x^{-3}$
6.	$y = x^{½}$	6.	$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$
7.	$y = \frac{1}{x^{10}}$	7.	$\frac{dy}{dx} = -10x^{-11}$
8.	$y = 2x + 12x^3$	8.	$\frac{dy}{dx} = 2 + 36x^2$
9.	$y = 2x^{-½} - x^{\frac{3}{2}}$	9.	$\frac{dy}{dx} = -x^{-\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}}$
10.	$y = \sqrt{x} + \frac{6}{x^4}$	10.	$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - 24x^{-5}$

Differentiation of ln x, exponentials, sine and cosine

ln x

- $y = \ln x, \quad \frac{dy}{dx} = \frac{1}{x}$
- $y = \ln kx$, where k is a constant
  $\frac{dy}{dx} = \frac{1}{x}$

e^x

- $y = e^x, \quad \frac{dy}{dx} = e^x$
- $y = e^{kx}$, where k is a constant
  $\frac{dy}{dx} = ke^{kx}$

Sin x

- $y = \sin x, \quad \frac{dy}{dx} = \cos x$
- $y = \sin (kx)$, where k is a constant
  $\frac{dy}{dx} = k\cos (kx)$

Cos x

- $y = \cos x, \quad \frac{dy}{dx} = -\sin x$
- $y = \cos (kx)$, where k is a constant
  $\frac{dy}{dx} = -k\sin (kx)$

Questions		**Solutions**	
1.	$y = \ln 6x$	1.	$\frac{dy}{dx} = \frac{1}{x}$
2.	$y = e^{9x}$	2.	$\frac{dy}{dx} = 9e^{9x}$
3.	$y = \sin(5x)$	3.	$\frac{dy}{dx} = 5\cos(5x)$
4.	$y = \cos(3x)$	4.	$\frac{dy}{dx} = -3\sin(3x)$
5.	$y = \ln 4x + e^{-2x} + \sin(-7x)$	5.	$\frac{dy}{dx} = \frac{1}{x} - 2e^{-2x} - 7\cos(-7x)$

Higher Order Derivatives

Sometimes it is necessary to differentiate a function more than once, for example, when identifying the nature of stationary points.

$$y = f(x)$$

First derivative	$\frac{dy}{dx} = f'(x)$
Second derivative	$\frac{d^2y}{dx^2} = f''(x)$
Third derivative	$\frac{d^3y}{dx^3} = f'''(x)$

Example: Find the first and second derivative of $y = 3x^5 + \sin(2x)$

Solution: $\frac{dy}{dx} = 15x^4 + 2\cos(2x)$; $\quad \frac{d^2y}{dx^2} = 60x^3 - 4\sin(2x)$

Question: Find the first, second and third derivative of $y = 2x^3 + \cos(3x)$

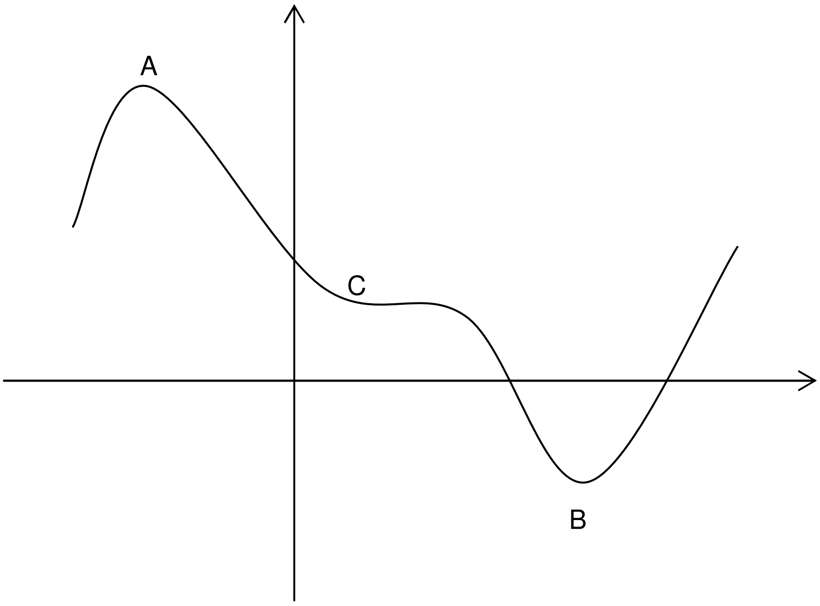
Solution: $\frac{dy}{dx} = 6x^2 - 3\sin(3x)$; $\quad \frac{d^2y}{dx^2} = 12x - 9\cos(3x)$

$$\frac{d^3y}{dx^3} = 12 + 27\sin(3x)$$

Maxima, Minima and Points of Inflexion

The derivative provides information about the gradient of the function. We can use this information to locate turning points on a graph, since at turning points the gradient is zero.

If the graph of our function is:



At the points A, B and C, the gradient is zero, hence we have three stationary points. This means that at these points $\frac{dy}{dx} = 0$.

Points A and B are referred to as turning points, since the curve turns. However point C is not a turning point since the graph is simply flat for a short time.

As well as being about to locate stationary points, it is also important to distinguish whether these are maximum turning points (point A), minimum turning points (point B) or points of inflexion (point C).

To classify turning points as maximum, minimum or points of inflexion, we need to find the second derivative of our function, i.e. $\frac{d^2y}{dx^2}$.

If

$$\frac{d^2y}{dx^2} > 0 \Rightarrow \text{minimum turning point}$$

$$\frac{d^2y}{dx^2} < 0 \Rightarrow \text{maximum turning point}$$

$$\frac{d^2y}{dx^2} = 0 \ \& \ \frac{d^3y}{dx^3} \neq 0 \Rightarrow \text{point of inflexion}$$

Example 1:

Find the turning point of the function $y = x^2 - 3x + 2$ and determine whether this is a maximum or minimum turning point.

Solution:

$y = x^2 - 3x + 2$

$$\frac{dy}{dx} = 2x - 3$$

We know that at a turning point $\frac{dy}{dx} = 0$, i.e.

$$2x - 3 = 0$$
$$2x = 3$$
$$x = \tfrac{3}{2}$$

To find the corresponding value of y, substitute $x = \tfrac{3}{2}$ into $y = x^2 - 3x + 2$

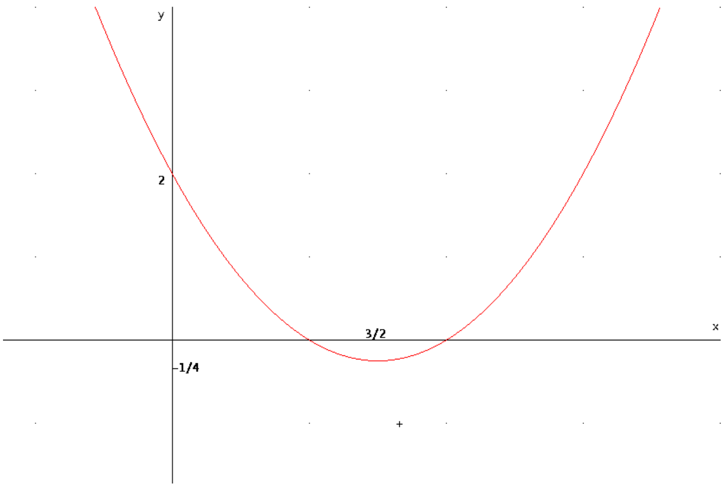
$$\begin{aligned} y &= x^2 - 3x + 2 \\ &= \left(\tfrac{3}{2}\right)^2 - 3\left(\tfrac{3}{2}\right) + 2 \\ &= -\tfrac{1}{4} \end{aligned}$$

Therefore there is a turning point at $\left(\tfrac{3}{2}, -\tfrac{1}{4}\right)$.

To distinguish whether this is a maximum or minimum turning point, find $\frac{d^2y}{dx^2}$:

$$\frac{d^2y}{dx^2} = 2 > 0 \Rightarrow \text{minimum turning point}$$

Therefore the turning point $\left(\tfrac{3}{2}, -\tfrac{1}{4}\right)$ is a minimum turning point.



Example 2:

Find the turning point of the function $y = x^3 - 3x + 2$ and hence determine whether this is a maximum or minimum turning point.

Solution:

$y = x^3 - 3x + 2$

$$\frac{dy}{dx} = 3x^2 - 3$$

We know that at a turning point $\frac{dy}{dx} = 0$, i.e.

$$3x^2 - 3 = 0$$
$$3x^2 = 3$$
$$x^2 = 1$$
$$x = \pm 1$$

Hence there are two turning points, one at $x = 1$ and another at $x = -1$.

When $x = 1$, $y = 1 - 3 + 2 = 0$

When $x = -1$, $y = (-1)^3 - 3(-1) + 2 = -1 + 3 + 2 = 4$

The two turning points are at (1, 0) and (-1, 4).

To distinguish whether these points are maximum or minimum turning points, find $\frac{d^2y}{dx^2}$:

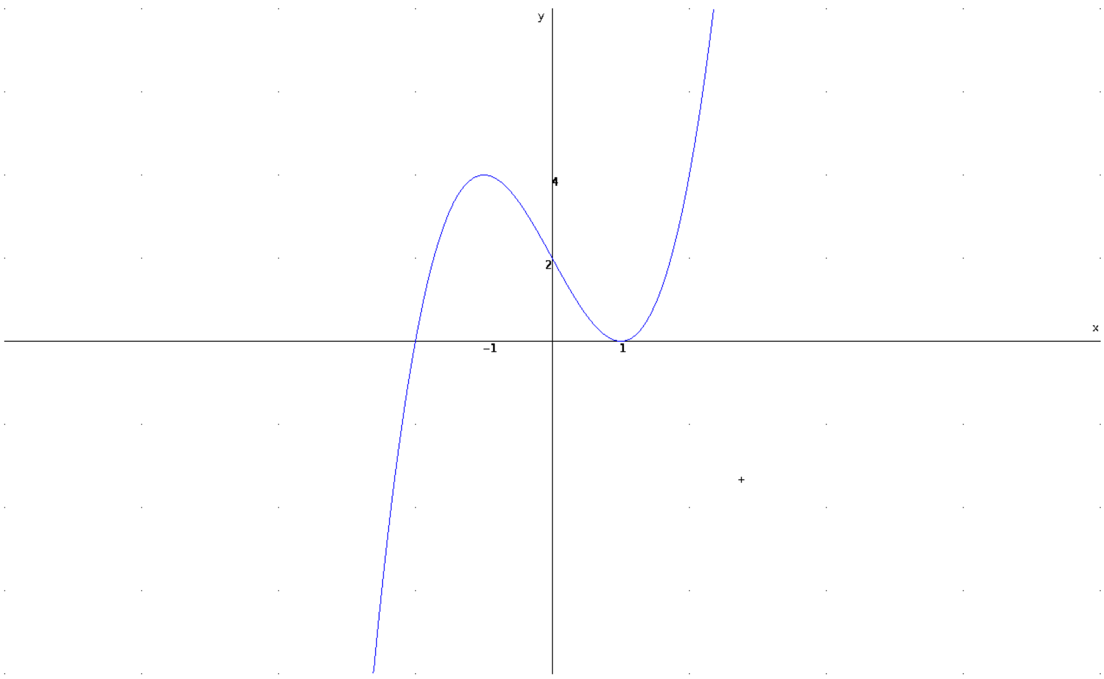
$$\frac{d^2y}{dx^2} = 6x$$

When x = 1, $\frac{d^2y}{dx^2} = 6 > 0 \Rightarrow$ minimum turning point

When x = -1, $\frac{d^2y}{dx^2} = -6 < 0 \Rightarrow$ maximum turning point

Therefore the point (1, 0) is a minimum turning point and (-1, 4) is a maximum turning point.

$y = x^3 - 3x + 2$



Example 3:

Find the turning point of the function $y = x^3 - 6x^2 + 9x + 6$ and hence determine whether this is a maximum or minimum turning point.

Solution:

$$y = x^3 - 6x^2 + 9x + 6$$

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$

We know that at a turning point $\frac{dy}{dx} = 0$, i.e.

$$3x^2 - 12x + 9 = 0$$
$$x^2 - 4x + 3 = 0$$
$$(x-1)(x-3) = 0$$
$$x = 1 \text{ and } x = 3$$

Hence there are two turning points, one at x = 1 and another at x = 3.

When x = 1, y = 10

When x = 3, y = 27 – 54 + 27 + 6 = 6

The two turning points are at (1, 10) and (3, 6).

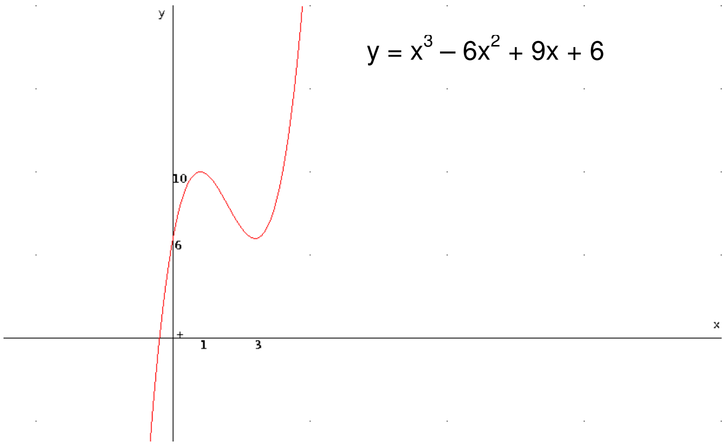
To distinguish whether these points are maximum or minimum turning points, find $\frac{d^2y}{dx^2}$:

$$\frac{d^2y}{dx^2} = 6x - 12$$

When x = 1, $\frac{d^2y}{dx^2} = -6 < 0 \Rightarrow$ maximum turning point

When x = 3, $\frac{d^2y}{dx^2} = 6 > 0 \Rightarrow$ minimum turning point

Therefore the point (1, 10) is a maximum turning point and (3, 6) is a minimum turning point.



Example 4:

Find the stationary point of the function $y = 2x^3 - 6x^2 + 6x - 5$ and classify the stationary point.

Solution:

$$y = 2x^3 - 6x^2 + 6x - 5$$

$$\frac{dy}{dx} = 6x^2 - 12x + 6$$

We know that at a turning point $\frac{dy}{dx} = 0$, i.e.

$$6x^2 - 12x + 6 = 0$$
$$x^2 - 2x + 1 = 0$$
$$(x - 1)(x - 1) = 0$$
$$x = 1$$

Hence there is one turning point at x = 1.

When x = 1, y = -3

To distinguish whether this point (1, -3) is a maximum, minimum or point of inflexion, find $\frac{d^2y}{dx^2}$:

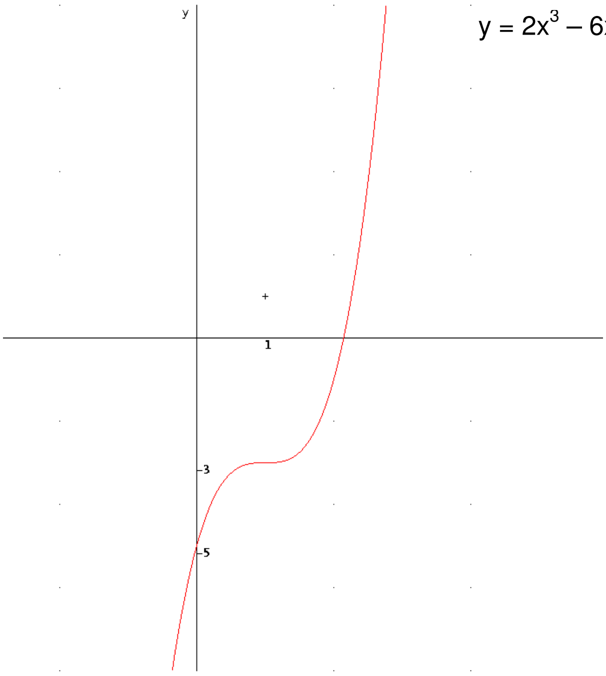
$$\frac{d^2y}{dx^2} = 12x - 12$$

When x = 1, $\frac{d^2y}{dx^2} = 0$

The point is neither a max or min, we therefore need to find $\frac{d^3y}{dx^3}$:

$$\frac{d^3y}{dx^3} = 12 \neq 0 \Rightarrow \text{point of inflexion}$$

Therefore the point (1, -3) is a point of inflexion.



$y = 2x^3 - 6x^2 + 6x - 5$

Questions

Find the turning points of the following functions and identify whether they are maximum or minimum turning points. Use this information to sketch the graph of each function.

1. $y = x^2 + 4x + 1$
2. $y = x^2 - 2x + 3$
3. $y = 2x^3 - 9x^2 + 12x$
4. $y = 2x^3 - 3x^2 - 6$
5. $y = x^4 + 4x^3 - 6$

Solutions

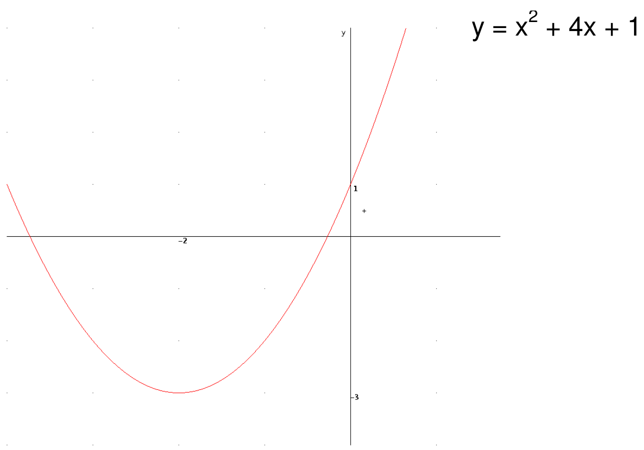
1. $y = x^2 + 4x + 1$

$$\frac{dy}{dx} = 2x + 4$$

Turning point occurs at (-2, -3)

$$\frac{d^2y}{dx^2} = 2$$

Therefore the turning point (-2, -3) is a minimum



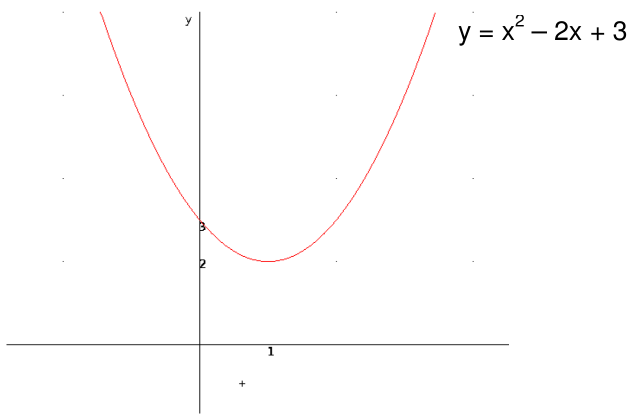
2. $y = x^2 – 2x + 3$

$$\frac{dy}{dx} = 2x – 2$$

Turning point occurs at (1, 2)

$$\frac{d^2y}{dx^2} = 2$$

Therefore the turning point (1, 2) is a minimum



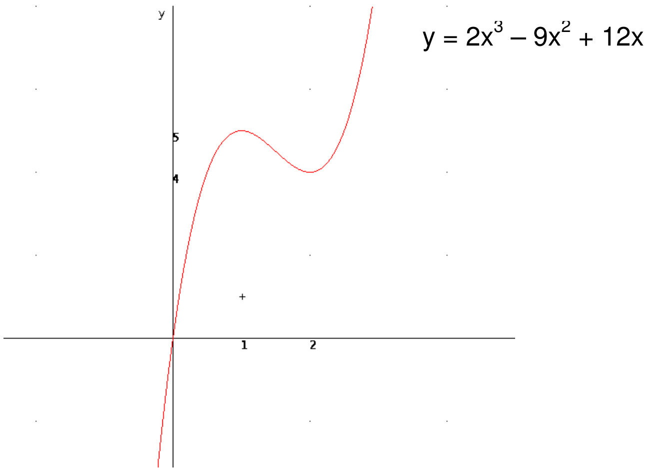
3. $y = 2x^3 - 9x^2 + 12x$

$$\frac{dy}{dx} = 6x^2 - 18x + 12$$

There are two turning points: (2, 4) and (1, 5)

$$\frac{d^2y}{dx^2} = 12x - 18$$

The turning point (2, 4) is a minimum and (1, 5) is a maximum turning point.



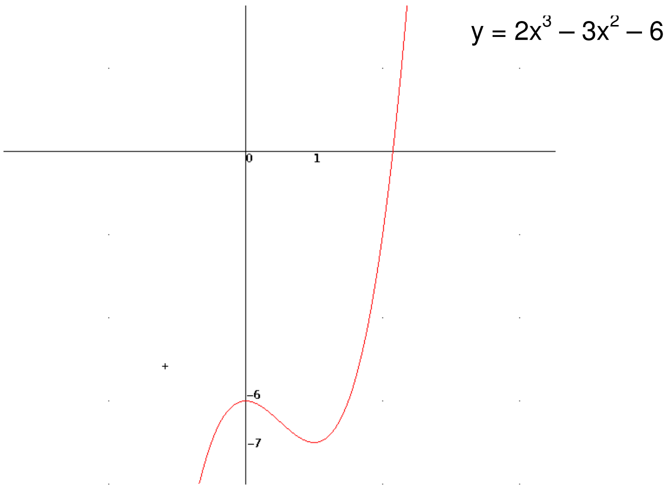
4. $y = 2x^3 - 3x^2 - 6$

$$\frac{dy}{dx} = 6x^2 - 6x$$

There are two turning points: (0, -6) and (1, -7)

$$\frac{d^2y}{dx^2} = 12x - 6$$

The turning point (0, -6) is a maximum and (1, -7) is a minimum turning point.



5. $y = x^4 + 4x^3 - 6$

$$\frac{dy}{dx} = 4x^3 + 12x^2$$

There are two turning points: (0, -6) and (-3, -33)

$$\frac{d^2y}{dx^2} = 12x^2 + 24x$$

When x = 0, $\frac{d^2y}{dx^2} = 0$, therefore need to find $\frac{d^3y}{dx^3}$:

$$\frac{d^3y}{dx^3} = 24x + 24$$

When x = 0, $\frac{d^3y}{dx^3} = 24 \neq 0$, therefore the point (0, -6) is a point of inflexion.

When x = -3, $\frac{d^2y}{dx^2} = 36 > 0 \Rightarrow$ minimum turning point

The turning point (0, -6) is a point of inflexion and (-3, -33) is a minimum turning point.

$y = x^4 + 4x^3 - 6$