Integration

Integration is used to find areas under curves.

Integration is the reversal of differentiation hence functions can be integrated by indentifying the anti-derivative.

However, we will learn the process of integration as a set of rules rather than identifying anti-derivatives.

Terminology

- **Indefinite** and **Definite** integrals

  There are two types of integrals: Indefinite and Definite.

  Indefinite integrals are those with no limits and definite integrals have limits.

  When dealing with indefinite integrals you need to add a constant of integration. For example, if integrating the function *f(x)* with respect to x:

  $$\int f(x)\mathrm{dx} = \mathrm{g(x)} + \mathrm{C}$$

  where g(x) is the integrated function.

- **C** is an arbitrary constant called the *constant of integration*.
- **dx** indicates the variable with respect to which we are integrating, in this case, *x*.
- The function being integrated, *f(x)*, is called the **integrand**.

The rules

- The Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \text{ provided that } n \neq -1$$

Examples: $\int x^5\,dx = \frac{x^6}{6} + C$

$$\int x^{-4}\,dx = \frac{x^{-3}}{-3} + C$$

- When n = -1

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln x + C$$

- Constant rule

$\int k\,dx = kx + C$ where k is a constant

Example: $\int 2\,dx = 2x + C$

- Exponentials

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$$

Example: $\int e^{9x}\,dx = \frac{1}{9}e^{9x} + C$

$$\int e^{x}\,dx = e^{x} + C$$

- Trig functions
  - Cos

$$\int \cos(x)\,dx = \sin(x) + C$$

$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + C$ where k is a constant

Example: $\int \cos(12x)\,dx = \frac{1}{12}\sin(12x) + C$

- Sin

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C \qquad \text{where k is a constant}$$

Example: $\int \sin(10x)\,dx = -\frac{1}{10}\cos(10x) + C$

$$\int \sin(-5x)\,dx = \frac{1}{5}\cos(-5x) + C$$

Linearity

Suppose *f(x)* and *g(x)* are two functions in terms of x, then:

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

Additionally, if A and B are constants, then

$$\int [Af(x) \pm Bg(x)]dx = A\int f(x)dx \pm B\int g(x)dx$$

Examples:

$$\int (2x^4 + 3x^5)dx = \int 2x^4\,dx + \int 3x^5\,dx$$

$$= 2\int x^4\,dx + 3\int x^5\,dx$$

$$= 2\left(\frac{x^5}{5}\right) + 3\left(\frac{x^6}{6}\right) + C$$

$$= \frac{2x^5}{5} + \frac{x^6}{2} + C$$

$$\int (5\cos(3x) - 3e^{7x})dx = \int 5\cos(3x)dx - \int 3e^{7x}\,dx$$

$$= 5\int \cos(3x)dx - 3\int e^{7x}\,dx$$

$$= 5\left(\frac{1}{3}\sin(3x)\right) - 3\left(\frac{1}{7}e^{7x}\right)$$

$$= \frac{5}{3}\sin(3x) - \frac{3}{7}e^{7x}$$

Questions (General rules):

Integrate the following functions:

1. $\int \left(x^6 - x^{\frac{3}{2}} + \frac{1}{x^5}\right)dx$
2. $\int \left(3x^8 + x - 5\right)dx$
3. $\int \left(9x^2 - 3x^{-1}\right)dx$
4. $\int \left(\sin(4x) + e^{3x}\right)dx$
5. $\int \left(\cos(7x) + 7x^2\right)dx$

(Solutions on page 8)

Definite Integrals

Earlier we saw that

$$\int f(x)dx = g(x) + C$$

Suppose now we are given limits, i.e.

$$\int_a^b f(x)dx = g(x) + C$$ (where a is the lower limit and b is the upper limit)

This can be interpreted as:

(value of g(x) + C at x = b) – (value of g(x) + C at x = a)

In other words, since C will cancel out:

$$\int_a^b f(x)dx = g(b) - g(a)$$

The full calculation of definite integrals is usually written out as:

$$\int_a^b f(x)dx = \left[g(x)\right]_a^b = g(b) - g(a)$$

i.e. integrate the function first (find g(x)) then substitute in the given limits (always substitute the upper limit first).

Examples

1. $\int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}\left[x^3\right]_0^1 = \frac{1}{3}\{(1)^3 - (0)^3\} = \frac{1}{3}(1 - 0) = \frac{1}{3}$

2. $\int_1^3 (2x+1)\,dx = \left[\frac{2x^2}{2} + x\right]_1^3 = \left[x^2 + x\right]_1^3 = \{(3^2 + 3) - (1^2 + 1)\}$

   $= \{(9 + 3) - (1 + 1)\} = 12 - 2 = 10$

3. $\int_0^{\frac{\pi}{2}} \cos(x)\,dx = \left[\sin(x)\right]_0^{\frac{\pi}{2}} = \{(\sin(\frac{\pi}{2})) - (\sin(0))\}$

   $= 1 - 0 = 1$

Questions (Definite integrals)**:**

Integrate the following functions:

1. $\int_1^2 (3x^2 - 2x + 5)\,dx$
2. $\int_0^1 e^{7x}\,dx$
3. $\int_0^{\pi} \sin(2x)\,dx$
4. $\int_1^4 (12e^{4x} + 4\sqrt{x})\,dx$

(Solutions on page 9)

Integration that leads to log functions

We know that if we differentiate y = ln(x) we find $\frac{dy}{dx} = \frac{1}{x}$.

We also know that if y = ln f(x), this differentiates as:

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

Example: $y = \ln(2x^2 + 5)$

$t = 2x^2 + 5$ $\quad$ $y = \ln t$

$\frac{dt}{dx} = 4x$ $\quad$ $\frac{dy}{dt} = \frac{1}{t}$

$\frac{dy}{dx} = 4x \times \frac{1}{t} = \frac{4x}{t} = \frac{4x}{2x^2 + 5}$

If we can recognise that the function we are trying to integrate is the derivative of another function, we can simply reverse the above process. So if the function we are trying to integrate is a quotient, and if the numerator is the derivative of the denominator, then the integral will involve a logarithm, i.e.

$$\int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) + C$$

Example 1: $\int \frac{5}{3+5x}dx$

The derivative of the denominator is 5 which is the same as the numerator, hence

$$\int \frac{5}{3+5x}dx = \ln(3 + 5x) + C$$

Example 2: $\int \frac{x}{1+x^2}dx$

The derivative of the denominator is 2x. This is not the same as the numerator but we can make it the same by re-writing the function

$\frac{x}{1+x^2}$ as $\frac{1}{2}\cdot\frac{2x}{1+x^2}$, therefore

$$\int \frac{x}{1+x^2}dx = \frac{1}{2}\int \frac{2x}{1+x^2}dx = \tfrac{1}{2}\ln(1 + x^2) + C$$

Example 3: $\int \frac{1}{x\ln(x)}dx$

The derivative of ln x is $\frac{1}{x}$, so we can rewrite the function as:

$\frac{{}^{1}\!/_{x}}{\ln(x)}$. Hence

$$\int \frac{1}{x\ln(x)}dx = \int \frac{{}^{1}\!/_{x}}{\ln(x)}dx = \ln(\ln(x)) + C$$

Example 4: $\int_1^2 \left(\frac{3}{x} - \frac{3}{x+1}\right)dx$

$$\int_1^2 \left(\frac{3}{x} - \frac{3}{x+1}\right)dx = 3\int_1^2 \left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= \left[3\ln(x) - 3\ln(x+1)\right]_1^2$$

$$= \{(3\ln(2) - 3\ln(3)) - (3\ln(1) - 3\ln(2))\}$$

$$= 3\ln(2) - 3\ln(3) + 3\ln(2) = 6\ln(2) - 3\ln(3)$$

$$= \ln(2^6) - \ln(3^3)$$

$$= \ln(64) - \ln(27) = \ln\left(\frac{64}{27}\right)$$

Questions (Integration that leads to log functions):

Integrate the following functions:

1. $\int \frac{3}{2+3x} dx$

2. $\int \frac{x}{1+2x^2} dx$

3. $\int \frac{e^{2x}}{e^{2x}+1} dx$

4. $\int \frac{x^{-3}}{x^{-2}+4} dx$

5. $\int_0^1 \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx$

(Solutions on page 10)

Solutions (General rules):

1. $$\int\left(x^6 - x^{\frac{3}{2}} + \tfrac{1}{x^5}\right)dx = \int x^6\,dx - \int x^{\frac{3}{2}}\,dx + \int \tfrac{1}{x^5}\,dx$$
$$= \int x^6\,dx - \int x^{\frac{3}{2}}\,dx + \int x^{-5}\,dx$$
$$= \frac{x^7}{7} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^{-4}}{(-4)} + C$$
$$= \frac{x^7}{7} - \frac{2x^{\frac{5}{2}}}{5} - \frac{1}{4x^4} + C$$

2. $$\int\left(3x^8 + x - 5\right)dx = \int 3x^8\,dx + \int x\,dx \ - \int 5\,dx$$
$$= 3\int x^8\,dx + \int x\,dx \ - \int 5\,dx$$
$$= \frac{3x^9}{9} + \frac{x^2}{2} - 5x + C$$
$$= \frac{x^9}{3} + \frac{x^2}{2} - 5x + C$$

3. $$\int\left(9x^2 - 3x^{-1}\right)dx = \int 9x^2\,dx - \int 3x^{-1}\,dx$$
$$= 9\int x^2\,dx - 3\int x^{-1}\,dx$$
$$= \frac{9x^3}{3} - 3\ln(x) + C$$
$$= 3x^3 - 3\ln(x) + C$$

4. $$\int\left(\sin(4x) + e^{3x}\right)dx = \int \sin(4x)\,dx + \int e^{3x}\,dx$$
$$= -\frac{1}{4}\cos(4x) + \frac{1}{3}e^{3x} + C$$

5. $$\int\left(\cos(7x) + 7x^2\right)dx = \int \cos(7x)\,dx + 7\int x^2\,dx$$
$$= \frac{1}{7}\sin(7x) + \frac{7}{3}x^3 + C$$

Solutions (Definite integrals):

1. $\int_1^2 (3x^2 - 2x + 5)dx = \left[\frac{3x^3}{3} - \frac{2x^2}{2} + 5x\right]_1^2$

$= [x^3 - x^2 + 5x]_1^2$

$= \{(2^3 - 2^2 + 5(2)) - (1^3 - 1^2 + 5(1))\}$

$= \{(8 - 4 + 10) - (1 - 1 + 5)\}$

$= 14 - 5$

$= 9$

2. $\int_0^1 e^{7x}\,dx = \left[\frac{1}{7}e^{7x}\right]_0^1 = \frac{1}{7}[e^{7x}]_0^1 = \frac{1}{7}\{e^7 - e^0\} = \frac{1}{7}(e^7 - 1)$

3. $\int_0^\pi \sin(2x)dx = \left[-\frac{1}{2}\cos(2x)\right]_0^\pi = -\frac{1}{2}[\cos(2x)]_0^\pi$

$= -\frac{1}{2}\{\cos(2\pi) - \cos(0)\}$

$= -\frac{1}{2}\{1 - 1\}$

$= 0$

4. $\int_1^4 \left(12e^{4x} + 4\sqrt{x}\right)dx = \int_1^4 \left(12e^{4x} + 4x^{\frac{1}{2}}\right)dx$

$= \left[\frac{12e^{4x}}{4} + \frac{4x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4 = \left[3e^{4x} + \frac{8x^{\frac{3}{2}}}{3}\right]_1^4$

$= \left\{\left(3e^{16} + \frac{8(4)^{\frac{3}{2}}}{3}\right) - \left(3e^4 + \frac{8}{3}\right)\right\}$

$= \left\{\left(3e^{16} + \frac{8(8)}{3}\right) - \left(3e^4 + \frac{8}{3}\right)\right\}$

$= 3e^{16} + \frac{64}{3} - 3e^4 - \frac{8}{3}$

$= 3e^{16} - 3e^4 + \frac{56}{3}$

Solutions (Integration that leads to log functions):

1. $\int \frac{3}{2+3x}dx = \ln(2+3x) + C$

2. $\int \frac{x}{1+2x^2}dx$

   Differentiating the denominator gives 4x

   Therefore rewrite the function:

   $$\frac{x}{1+2x^2} = \frac{1}{4}\cdot\frac{4x}{1+2x^2}$$

   Hence,

   $$\int \frac{x}{1+2x^2}dx = \int \frac{1}{4}\cdot\frac{4x}{1+2x^2}dx = \tfrac{1}{4}\int \frac{4x}{1+2x^2}dx = \tfrac{1}{4}\ln(1+2x^2) + C$$

3. $\int \frac{e^{2x}}{e^{2x}+1}dx$

   Differentiating the denominator gives $2e^{2x}$ hence we can rewrite the function as:

   $$\frac{e^{2x}}{e^{2x}+1} = \frac{1}{2}\cdot\frac{2e^{2x}}{e^{2x}+1}$$

   $$\int \frac{1}{2}\cdot\frac{e^{2x}}{e^{2x}+1}dx = \tfrac{1}{2}\ln(e^{2x}+1) + C$$

4. $\int \frac{x^{-3}}{x^{-2}+4}dx$

   Differentiating the denominator gives $-2x^{-3}$, hence the function can be rewritten as:

   $$\frac{x^{-3}}{x^{-2}+4} = -\frac{1}{2}\cdot\frac{2x^{-3}}{x^{-2}+4}$$

   $$\int \frac{x^{-3}}{x^{-2}+4}dx = -\frac{1}{2}\int \frac{2x^{-3}}{x^{-2}+4}dx = -\tfrac{1}{2}\ln(x^{-2}+4) + C$$

5. $\int_0^1 \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx = \left[\ln(x+1) - \ln(x+2) \right]_0^1$

$$= \{(\ln(2) - \ln(3)) - (\ln(1) - \ln(2))\}$$

$$= \ln(2) - \ln(3) + \ln(2)$$

$$= 2\ln(2) - \ln(3)$$

$$= \ln(2^2) - \ln(3)$$

$$= \ln(4) - \ln(3)$$

$$= \ln\left(\frac{4}{3} \right)$$