Workshop: Algebra 1

Topics Covered:

- Rules of indices
- Expanding bracketed expressions
- Factorisation of simple algebraic expressions

Powers

A power is used when a number is multiplied by itself several times.

For example:

2^3 → power
→ base

is interpreted as *'base 2 to the power of 3'*, i.e. $2 \times 2 \times 2$ (= 8). So 2 is being multiplied by itself three times.

The term power is more often referred to as **index**.

(NB: Plural of index is indices)

Key Points:

- Any number (or base) raised to the power of 1 is itself, e.g. $2^1 = 2$.
  More generally, $a^1 = a$.
- Any number raised to the power of 0 is always 1, e.g. $2^0 = 1$.
  More generally, $a^0 = 1$.

Reasoning:

$$\frac{a^m}{a^m} = a^{m-m} = a^0$$

However, dividing a number by itself gives 1, therefore, $a^0 = 1$.

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The rules of indices

The rules of indices are used to manipulate algebraic and numerical expressions. These rules can only be applied to terms that have the same base.

1. $\boxed{a^m \times a^n = a^{m+n}}$

Examples:

$$2^2 \times 2^4 = 2^6$$

$$x^5 \times x^{10} = x^{15}$$

$$2xy \times 3x^2y^4 = 6x^3y^5$$

2. $\boxed{a^m \div a^n = a^{m-n}}$ $\boxed{\frac{a^m}{a^n} = a^{m-n}}$

Examples:

$$\frac{3^7}{3^2} = 3^5$$

$$\frac{a^{20}}{a^{17}} = a^3$$

$$\frac{15x^2y^9}{5xy^5} = 3xy^4$$

3. $\boxed{(a^m)^n = a^{mn}}$

Examples:

$$(6^5)^3 = 6^{15}$$

This rule (rule 3), can also be used to manipulate expressions such as:

$$\boxed{(a^m b^n)^k = a^{mk} b^{nk}}$$

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For example,

$$(a^3b^2)^4 = a^{12}b^8$$

$$(2x^5y)^4 = 16x^{20}y^4$$

Negative Powers:

Theory:

Let's look at $\frac{a^m}{a^{m+n}}\left(=\frac{a^m}{a^m a^n}=\frac{1}{a^n}\right)$, clearly this is the same as $\frac{1}{a^n}$, using rule 2, this can be re-written as: $a^{m-(m+n)} = a^{-n}$.

Therefore, when a number is raised to a negative power, e.g. a^{-m}, it can be rewritten as: $\frac{1}{a^m}$, i.e.

$$\boxed{a^{-m} = \mathbf{1/a^m}}$$

Example: $4^{-2} = \frac{1}{4^2}$

Roots: A basic root that you should know is $x^{\frac{1}{2}} = \sqrt{x}$.

In general,

Positive integer

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$ (where n is a natural number)

Taking this further,

$$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{x}\right)^m.$$

Examples:

$$4^{\frac{1}{2}} = \sqrt{4} = 2$$

$$27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{27}\right)^2 = (3)^2 = 9$$

Questions (Rules of Indices)**:**

Simplify the following:

1. $a^{-2} \times a^6$ **2.** $3x^6 \times 5x^9$ **3.** $(a^2bc^2)(ab^2c)$

4. $\frac{5^3}{5^2}$ **5.** $\frac{6x^2y^3}{3xy^2}$ **6.** $(a^{-2})^5$

7. $(3x^2y^3)^4$ **8.** $(x^{-2}y^5)^3$ **9.** $64^{\frac{1}{3}}$

10. $16^{\frac{3}{2}}$ **11.** $125^{-\frac{2}{3}}$

(Solutions on page 8)

Expanding Bracketed Expressions

When brackets are removed (or expanded) from around an algebraic expression, any factors outside the brackets must be multiplied to each term inside the brackets.

For example,

$$a(x + y)$$

'a' is the factor and is directly outside the brackets. Therefore, when expanding (or removing) the brackets, the factor 'a' has to be multiplied to both 'x' and 'y' inside the brackets, i.e.

$$a(x + y) = ax + ay$$

In a similar way, if the factor outside the brackets is a negative number/expression, the negative factor has to be multiplied to every term within the brackets, i.e.

$$-a(x + y) = -ax - ay$$

In the same way,

$$a(x - y) = ax - ay$$

$$-a(x - y) = -ax + ay$$

Examples:

1. $2ab(3a + bc) = 6a^2b + 2ab^2c$
2. $-5x^2y(2xy - 4x + 3) = -10x^3y^2 + 20x^3y - 15x^2y$
3. $3xy^2z(2x + 4xz - 5y^3z^2 + 6) = 6x^2y^2z + 12x^2y^2z^2 - 15xy^5z^3 + 18xy^2z$

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Questions (Expanding bracketed expressions)**:**

Remove the brackets from the following bracketed expressions:

1. $5x^2(4 - x)$
2. $-4a(2a - 5b + 4ac)$
3. $6(y + 2) - 3y$
4. $3(5x - y) + 2(2x - 6y)$

(Solutions on page 8)

Multiplying together two bracketed terms

Method 1: Smiley face

$$(a + b)(c + d)$$

Method 2: FOIL (**F**irst, **O**uter, **I**nner, **L**ast)

F(irst) The product of the two first terms in each bracket

O(uter) The product of the two outer terms

I(nner) The product of the two inner terms

L(ast) The product of the two last terms in each bracket

$$(a + b)(c + d)$$

Some generic expansions:

$$(a + b)^2 = (a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$$

$$\left.\begin{aligned}(a + b)(a - b) &= a^2 - ab + ab - b^2 = a^2 - b^2 \\ (a - b)(a + b) &= a^2 + ab - ab - b^2 = a^2 - b^2\end{aligned}\right\} \text{Difference of squares}$$

$$(a - b)(a - b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$$

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Examples:

1. $(x + 5)(x - 2) = x^2 - 2x + 5x - 10 = x^2 + 3x - 10$
2. $(x - 3)(y + 5) = xy + 5x - 3y - 15$
3. $(2x - 6)(x - 2) = 2x^2 - 4x - 6x + 12 = 2x^2 - 10x + 12$
4. $(4x + 5)(2x^2 + 3y) = 8x^3 + 12xy + 10x^2 + 15y$

Questions (Multiplying out two brackets)**:**

Expand the following:

1. $(x - 4)(x + 5)$
2. $(2x + 7)(x - 2)$
3. $(3x - 2)(5x - 1)$
4. $(3x - 2y)(4x + 5y)$
5. $(x + 1)(x^2 - 5x^3 + 4)$

(Solutions on page 8)

Factorisation of Algebraic Expressions

Factorisation is the reversal of expanding brackets.

Common factors are identified and taken outside a bracketed term.

You can always check whether your factorised expression is correct by expanding/removing the brackets and checking you get the original expression.

$a(b + c) = ab + ac$

$a(b - c) = ab - ac$

$-a(b + c) = -ab - ac$

$-a(b - c) = -ab + ac$

$(a + b)(a + b) = a^2 + 2ab + b^2$

$(a - b)(a - b) = a^2 - 2ab + b^2$

$(a + b)(a - b) = a^2 - b^2$

Examples:

1. $4x + 12 = 4(x + 3)$
2. $8x^2 - 16x = 8x(x - 2)$
3. $10x^3 + 15x^2 = 5x^2(2x + 3)$
4. $9x^2 - 3x + 6xy = 3x(3x - 1 + 2y)$

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Questions (Factorisation of algebraic expressions)**:**

Factorise the following algebraic expressions

1. $7x + 21$
2. $25 - 10y$
3. $9x^2 + 3x$
4. $4x^2 - 20xy$
5. $5x^3 + 10x^2$
6. $36y^4 - 6y + 18y^2$
7. $8x^4y + 10x^2y^2 - 6x^2y^3$
8. $12x^2y^3 + 6x^5$

(Solutions on page 8)

Solutions (Rules of Indices)**:**

1. a^4 **2.** $15x^{15}$ **3.** $a^3b^3c^3$ **4.** $5^1 = 5$

5. $2xy$ **6.** a^{-10} **7.** $81x^8y^{12}$ **8.** $x^{-6}y^{15}$

9. $64^{\frac{1}{3}} = \sqrt[3]{64} = 4$

10. $16^{\frac{3}{2}} = \left(16^{\frac{1}{2}}\right)^3 = (4)^3 = 64$

11. $125^{-\frac{2}{3}} = \dfrac{1}{125^{\frac{2}{3}}} = \dfrac{1}{\left(125^{\frac{1}{3}}\right)^2} = \dfrac{1}{(5)^2} = \dfrac{1}{25}$

Solutions (Expanding bracketed expressions)**:**

1. $5x^2(4 - x) = 20x^2 - 5x^3$
2. $-4a(2a - 5b + 4ac) = -8a^2 + 20ab - 16a^2c$
3. $6(y + 2) - 3y = 6y + 12 - 3y = 3y + 12$
4. $3(5x - y) + 2(2x - 6y) = 15x - 3y + 4x - 12y = 19x - 15y$

Solutions (Multiplying out two brackets)**:**

1. $(x - 4)(x + 5) = x^2 - 4x + 5x - 20 = x^2 + x - 20$
2. $(2x + 7)(x - 2) = 2x^2 - 4x + 7x - 14 = 2x^2 + 3x - 14$
3. $(3x - 2)(5x - 1) = 15x^2 - 3x - 10x + 2 = 15x^2 - 13x + 2$
4. $(3x - 2y)(4x + 5y) = 12x^2 + 15xy - 8xy - 10y^2$
   $= 12x^2 + 7xy - 10y^2$
5. $(x + 1)(x^2 - 5x^3 + 4) = x^3 - 5x^4 + 4x + x^2 - 5x^3 + 4$
   $= -5x^4 - 4x^3 + x^2 + 4x + 4$

Solutions (Factorisation of algebraic expressions)**:**

1. $7(x + 3)$
2. $5(5 - 2y)$
3. $3x(3x + 1)$
4. $4x(x - 5y)$
5. $5x^2(x + 2)$
6. $6y(6y^3 - 1 + 3y)$
7. $2x^2y(4x^2 + 5y - 3y^2)$
8. $6x^2(2y^3 + x^3)$

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